

## The thermodynamically consistent equations of a thermoelastic saturated porous medium<sup>☆</sup>

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### Abstract

A phenomenological model of a porous medium saturated with fluid is considered within the framework of the hypothesis of interpenetrating continua. Assuming that there are no phase transitions, that the contribution of pulsations to the stress tensor and kinetic energy is small, and the components of the medium are in thermal equilibrium, mass, momentum and energy equations and a law of conservation of compatibility of the deformations and velocities are formulated. Using a representation of the force of interaction of the components in the form of the sum of equilibrium and dissipative components, a new form of inequality is obtained for the rate of entropy production. A definition of a thermoelastic saturated porous medium is given. The symmetry group of such a medium is considered as a set of two groups, corresponding to the symmetry of the skeleton and the fluid. It is shown that, in the class of thermoelastic porous media with an arbitrary type of symmetry of the skeleton, the saturating fluid can only be an ideal fluid, while the thermodynamic potentials and the porosity, stresses and entropies determined by them do not depend on the temperature gradient and the relative fluid velocity. It is found that the condition of incompressibility of only one of the components of the medium leads to the elimination of the porosity from the governing relations, rather than to kinematic limitations. The limitations imposed on the governing relations by the principle of thermodynamic consistency and the requirement of independence of the choice of the frame of reference are investigated. A form of the governing relations, necessary and sufficient to satisfy these principles, is obtained. It is shown that the Biot equations are one of the forms of thermodynamically consistent governing relations. A thermodynamic validation of the effective-stress tensor is given.

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From the Editorial Board. Professor Vladimir Ignat'evich Kondaurov, Doctor of Physico-Mathematical Sciences, Director of the "Continuum Mechanics" specialization at Moscow Physico-Technical Institute (State University), was 60 years old on 8 March 2007. His main areas of scientific activity are as follows: the mechanics of diffuse fracture, the theory of phase transitions of solids, and modelling of dynamic processes in elastoplastic media. He formulated a new law of conservation of the compatibility of finite deformations and velocities, and proposed an analogue of the Clausius - Clapeyron equations for solid-phase transitions. He developed a new approach to the description of the continual fracture of solids, which takes into account explicitly the irreversible transformation of the elastic energy into surface energy of microcracks. In recent years his scientific interests have been concentrated in the area of the mechanics and thermodynamics of saturated porous media. He has published more than 130 works, among which are four monographs. He is a member of the Russian National Committee on Theoretical and Applied Mechanics,

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a member of the Expert Council on Earth Sciences of the Russian Foundation for Basic Research and a number of committees on deformable solid mechanics.

The editorial board and staff of PMM, his colleagues and students congratulate him on his jubilee, and wish him good health and new scientific achievements.

One can distinguish two main directions in the mathematical theory of a saturated porous medium: 1) micromechanical, which uses the procedure of averaging the equations for each component of a heterogeneous mixture of two materials, and 2) phenomenological, which investigates the limitations imposed on the equations of state, which arise from general principles of the theory of governing equations (independence of the choice of the frame of reference, thermodynamic consistency, and invariance with respect to transformations of the symmetry of the reference configurations). Despite undoubted successes, research in both areas is far from complete.

Below, within the framework of the second area of investigation, we consider a model of a thermoelastic saturated porous medium for finite deformations and arbitrary changes in the temperature. The saturated porous medium is regarded as a combination of two continua – a skeleton and a fluid which saturates it, the elements of which can be simultaneously situated at the point in space considered. Successive discussion of the macroscopic description of a saturated porous medium was apparently given for the first time by Biot;<sup>1</sup> reviews of the present state of research in this area can be found in Refs 2–4.

A description of the kinematics of the macroscopic motion of several continua having a certain novelty of approach is given. A central place is occupied by the equations of compatibility of finite deformations and velocities, which are useful for describing many problems of the non-linear mechanics of a saturated porous medium. These equations are extremely promising for investigating the general properties. They enable one to write a complete system of equations in divergent form, to obtain a closed system of relations at strong discontinuities, to develop numerical methods, conservative with respect to all the components of the vector of the solution, and to reduce the system of equations to symmetrical form, and on the basis of this, to obtain certain a priori limitations, imposed on the thermodynamic potentials of a saturated porous medium.

Unlike the majority of models, in which, as a rule, the effect of the reference (initial) states of the components of the saturated porous medium on the form of the governing relations is ignored, here considerable attention is devoted to this problem. The idea of a symmetry group of a saturated porous medium as a combination of two groups of transformations of the initial states of the skeleton and the fluid, which leave the governing equations unchanged, is introduced. This approach enables the symmetry group of a porous medium, saturated with several fluids, to be determined in a natural way, and reveals the structure of the governing equations for the case of an arbitrary anisotropy of the skeleton.

A step-by-step determination of the state and response of a thermoelastic saturated porous medium with viscous interaction of the components is given. Quite unexpectedly, this turned out to be not a trivial problem. In fact, attempts to consider the governing relation for the interaction force encounter the fact that this force, in a state of equilibrium, is proportional to the porosity gradient,<sup>3,5</sup> which itself is a function of the state. Consequently, the interaction force depends on the gradients of the parameters of state. Subsequent consideration of this fact of necessity leads to a micropolar theory with an asymmetrical stress tensor. Echoes of these difficulties can be found, for example, in Ref. 6.

A way out can be obtained by representing the interaction force of the components of the saturated porous medium in the form of the sum of equilibrium and dissipative components. A governing relation is postulated for the dissipative term, whereas the equilibrium part is described by a universal expression, which is independent of the rheological properties of the fluid. Expansion of the interaction force required the introduction of an equilibrium porosity and an equilibrium stress tensor of the fluid, which, it would appear, was previously not necessary to consider for thermoelastic bodies. In turn, this implied a new formulation of the inequalities for the rate of entropy production, in which these equilibrium quantities were explicitly used. This inequality enabled a number of consequences to be obtained, the major one of which was previously known either in the form of sufficient conditions or additional assumptions.

New answers are given to some problems of constructing the form of the inequality for the rate of entropy production. As an example we can mention the dependence of the response on the deformations of all the components of the saturated porous medium. To solve this problem, some authors introduce, either in explicit<sup>7</sup> or implicit form<sup>5</sup>, a so-called principle of separation of the phases. This essentially consists of the a priori requirement that the thermodynamic potential, the stress and the entropy of any component should depend only on the variables characterizing this particular component, while the heat fluxes, the interaction forces and the porosity must be determined by all the independent variables. It is shown that, generally speaking, this principle is not true, since the condition of thermodynamic compatibility leads to

the fact that the potential, the threshold pressure and the entropy of the fluid depend on its temperature and true density, while for the skeleton they depend on the parameters of the state of both continua.

It is shown that the equipresence rule, well known in classical continuum mechanics (the requirement that the initial set of parameters of state should be the same for all the governing equations) also operates in the material considered. By virtue of this rule, the temperature gradient, being “natural” for heat flux, and the relative velocity – for the interaction force, are also included in a number of the initial arguments of all the defining functions. The requirement of thermodynamic consistency excludes the temperature gradient and the relative velocity from the thermodynamic potential and the stresses and entropy defined by them, but the heat flux and the interaction forces depend on both of these quantities, demonstrating the effect of the heat and mass fluxes in the case of large deformations and finite temperature changes.

In models of a saturated porous medium with an incompressible skeleton, the concept of an effective stress, attributed to Terzaghi,<sup>8</sup> is widely used, and is the basis of numerous subsequent papers.<sup>9</sup> The intuitive introduction of this tensor, equal to the sum of the total stress and the spherical tensor  $-p\mathbf{I}$ , where  $p$  is the threshold pressure of the fluid, is validated, in the best case, by the presence of an additional relation – the condition that the skeleton is incompressible.<sup>4</sup> It should, however, be noted that the kinematic limitations of the type in which the divergence of the velocity is equal to zero, which enables one to add to the stress tensor any spherical tensor which does not perform work on such velocity fields, only occur for an undeformable skeleton. In other cases the divergence of the velocity is non-zero. The principle of thermodynamic consistency shows that the effective stress is produced by a specific dependence of the divergence of the seepage rate on the rate of change of distortion of the skeleton.

In the case of a compressible skeleton, the key fact<sup>1</sup> is the relation between the total-stress tensor and the potential representing the Legendre transformation of the free energy of the skeleton. But not a single paper, as far as I am aware, answers the question – are these relations a special case of more general equations of state of a thermoelastic saturated porous medium or do they include all valid models of such media? Investigation shows that, in the class of thermoelastic saturated porous media, for which the contribution of pulsations to the stress and kinetic energy is small, the Biot equations are unimprovable, being the necessary and sufficient conditions for independence of the choice of the reference frame and of thermodynamic consistency. But these equations are not the only form of such governing relations, and systems of equations equivalent to them, possessing the same properties, exist.

## 1. The kinematics of a saturated porous medium

In the case of a porous medium, made up of several continua, saturated with fluid, which simultaneously occupy a certain region of space, the Euler method of describing the motion is the most convenient. Suppose  $\chi$  is a fixed region in three-dimensional Euclidean space. Elements of the skeleton and fluid are combined at each point  $\mathbf{x} \in \chi$ . The regions  $\kappa_A(t)$ ,  $A = S, F$ , occupied, at the initial instant of time, by particles of the skeleton and fluid and which form the configuration  $\chi$  at the instant  $t$ , will be called the reference (initial) configuration of the skeleton and the fluid respectively. Here and henceforth the subscript  $F$  denotes the fluid and the subscript  $S$  denotes the skeleton. The time-dependence of the regions  $\kappa_A(t)$  emphasises the fact that, at different instants of time, in the region  $\chi$  there are, in general, different combinations of particles of the skeleton and fluid. Points of these regions will be denoted by the symbol  $\mathbf{X}_A \in \kappa_A(t)$ .

The mappings  $\kappa_A(t) \rightarrow \chi$  will be assumed to be piecewise-homeomorphic and differentiable. We then have the following one-to-one differentiable laws of motion

$$\mathbf{X}_A = \mathbf{X}_A(\mathbf{x}, t), \quad \mathbf{x} \in \chi, \quad \mathbf{X}_A \in \kappa_A(t), \quad t \geq 0, \quad A = S, F \quad (1.1)$$

Suppose the asymmetric second-rank tensor  $\mathbf{G}_A = [\nabla \otimes \mathbf{X}_A(\mathbf{x}, t)]^T$  is the distortion of the continuum  $A$ , which represents the gradient of the mapping  $\kappa_A(t) \rightarrow \chi$ , so that  $d\mathbf{X}_A = \mathbf{G}_A \cdot d\mathbf{x}$ . As a consequence of the one-to-one nature of the mappings (1.1), the tensors  $\mathbf{G}_A$  are non-degenerate, i.e.  $J_A = \det \mathbf{G}_A \neq 0$ . For the non-degenerate tensor  $\mathbf{G}_A$  we have the following polar expansion<sup>10</sup>

$$\mathbf{G}_A = \mathbf{R}_A \cdot \mathbf{U}_A = \mathbf{V}_R \cdot \mathbf{R}_A \quad (1.2)$$

where  $\mathbf{U}_A$  and  $\mathbf{V}_A$  are symmetrical positive-definite tensors, and  $\mathbf{R}_A$  is an orthogonal tensor. These tensors characterize the dilatation and rotation of the neighbourhood of the point  $\mathbf{x}$ , at which the particle  $\mathbf{X}_A(\mathbf{x}, t)$  is situated. Expansions

(1.2) possess the property of uniqueness. In applied investigations, it is more convenient to use Lagrange and Almansi symmetrical finite-deformation tensors

$$\mathbf{E}_A = (\mathbf{G}_A^{-1T} \cdot \mathbf{G}_A^{-1} - \mathbf{I})/2, \quad \mathbf{A}_A = (\mathbf{I} - \mathbf{G}_A^T \cdot \mathbf{G}_A)/2 \tag{1.3}$$

which are equal to zero for motions of the continuum as a rigid whole, when  $\mathbf{G}_A$  is an orthogonal tensor.

We will denote by  $\mathbf{v}_A = \partial \mathbf{x}(\mathbf{X}_A, t)/\partial t|_{\mathbf{X}_A}$ ,  $\mathbf{M}_A = \partial \mathbf{X}_A(\mathbf{x}, t)/\partial t|_{\mathbf{x}}$  the mass velocities and the velocities of displacement of the particles of component  $A$  of the medium considered at the point  $\mathbf{x}$ . Differentiating with respect to  $t$  with fixed  $\mathbf{X}_A$  the laws of motion (1.1), written in the form of the complex function  $\mathbf{X}_A = \mathbf{X}_A(\mathbf{x}(\mathbf{X}_A, t), t)$ , we arrive at the relation  $\mathbf{M}_A = -\mathbf{G}_A \cdot \mathbf{v}_A$ .

The distortion  $\mathbf{G}_A(\mathbf{x}, t)$  and the velocity  $\mathbf{M}_A(\mathbf{x}, t)$  by definition are first derivatives of the mapping (1.1), and hence the following relation exists between them – the equation of compatibility of the deformations and velocities<sup>11</sup>

$$\partial \mathbf{G}_A^T / \partial t + \nabla \otimes (\mathbf{G}_A \cdot \mathbf{v}_A) = 0, \quad \text{or} \quad \partial \mathbf{G}_A^T / \partial t + \nabla \cdot (\mathbf{I} \otimes \mathbf{G}_A \cdot \mathbf{v}_A) = 0$$

Unless otherwise stated, there is no summation over the subscript  $A$ , which denotes that a quantity belongs to the component  $A$ .

Denoting the material derivative along the trajectory of a particle of the continuum  $A$  by

$$d_A f / dt \equiv \partial f(\mathbf{X}_A, t) / \partial t|_{\mathbf{X}_A} = \partial f(\mathbf{x}, t) / \partial t|_{\mathbf{x}} + \mathbf{v}_A \cdot \nabla f$$

and taking into account the relation  $\nabla_i G_{Ajk} = \nabla_k G_{Aji}$ , which expresses the possibility of a change in the order of differentiation with respect to the spatial variables  $x^i$  and  $x^k$ , we arrive at the equation of compatibility, written in the form

$$d_A \mathbf{G}_A / dt = -\mathbf{G}_A \cdot (\nabla \otimes \mathbf{v}_A)^T \tag{1.4}$$

Eq. (1.4) are formulated using the distortion, which includes the extension and rotation tensors of the material element. In cases when it is not necessary to calculate the rotations, the equations of compatibility, formulated in terms of “pure” strain are preferable. One of the possible equations of this type is the relation which connects the derivative of the Almansi tensor and the symmetrical tensor of deformation

$$D_A \mathbf{A}_A / Dt \equiv d_A \mathbf{A}_A / dt + \mathbf{A}_A \cdot (\nabla \otimes \mathbf{v}_A)^T + (\nabla \otimes \mathbf{v}_A) \cdot \mathbf{A}_A = (\nabla \otimes \mathbf{v}_A + (\nabla \otimes \mathbf{v}_A)^T) / 2$$

where  $D_A \mathbf{A}_A / Dt$  is the Oldroid derivative for the component  $A$ . However, this equation is not, in principle, reducible to divergent form and, consequently, is unsuitable for determining a weak (generalized) solution, for formulating relations on discontinuities, and for constructing conservative numerical methods.<sup>12</sup>

Taking into account Eq. (1.4) and the relations  $\partial J_A / \partial \mathbf{G}_A = J_A \mathbf{G}_A^{-1T}$  we can obtain an expression for the derivative of the determinant

$$d_A J_A / dt = -J_A \nabla \cdot \mathbf{v}_A \tag{1.5}$$

## 2. Conservation laws

We will confine ourselves to the case when there is no mass exchange between the continua (a phase or chemical transition). The mass conservation law for each of the components of the material can then be written in the form

$$\rho_A = \rho_{\kappa_A} J_A, \quad J_A = \det \mathbf{G}_A, \quad A = S, F \tag{2.1}$$

where  $\rho_A$  and  $\rho_{\kappa_A}$  are the average density of component  $A$  in the actual and initial states. Henceforth, in addition to the average density of the fluid  $\rho_F$  we will also use the true density  $\rho_f$ , which is introduced by the relation  $\rho_F = \phi \rho_f$ , where  $\phi(\mathbf{x}, t)$  is the volume porosity, equal to the fraction of the elementary volume occupied by the fluid. We will assume that the surface porosity or “transillumination”, equal to the fraction of the area occupied by the fluid in an arbitrary section of elementary volume,<sup>3</sup> is identical with the volume porosity. The sufficient condition for this is, for example, the condition of representativeness in combination with the boundedness of the “transillumination” gradient.<sup>5</sup>

If we use formula (1.5), we obtain from relation (2.1) the local equation of mass balance, or the continuity equation for the continuum  $A$

$$\partial \rho_A / \partial t + \nabla \cdot (\rho_A \mathbf{v}_A) = 0, \quad A = S, F \quad (2.2)$$

Eq. (2.2) can be obtained by averaging the continuity equation for the skeleton and the fluid written on the mesolevel. If we mean by the macroscopic velocity the mean-mass value of this quantity, pulsations of the quantities will cancel one another on averaging.<sup>5</sup>

In a number of cases, instead of the variable  $\mathbf{x}$  it is more convenient to use Lagrange variables, related to the particles of the skeleton. This is due to the fact that, in many applied problems, the skeleton is assumed to be a slightly deformed (or absolutely rigid) material, for which the reference configuration of the element considered differs little from the actual configuration (or is identical with it). In the variables  $\mathbf{X}_S$  the law of conservation of mass of the fluid has the form

$$\left. \frac{\partial \rho_F J_S^{-1}}{\partial t} \right|_{\mathbf{x}_S} + \overset{\circ}{\nabla} \cdot (\rho_F J_S^{-1} \mathbf{G}_S \cdot \mathbf{w}) = 0, \quad \text{or} \quad \frac{d_S(\rho_F J_S^{-1})}{dt} + J_S^{-1} \nabla \cdot (\rho_F \mathbf{w}) = 0 \quad (2.3)$$

where  $\overset{\circ}{\nabla}$  is the gradient in the variables  $\mathbf{X}_S$ ,  $\mathbf{w} = \mathbf{v}_F - \mathbf{v}_S$  is the relative velocity of the fluid, and  $J_S = \det \mathbf{G}_S$ . We can convince ourselves of the correctness of Eq. (2.3) by a direct check, if we use the definition of the derivative  $d_S/dt$ , kinematic relation (1.4), formula (1.5) and the Piola identity<sup>13</sup>  $\overset{\circ}{\nabla} \cdot (J_S^{-1} \mathbf{G}_S) = 0$ .

It is assumed, with regard to the force  $\mathbf{f}_A$ , acting on the continuum  $A$ , that it is equal to the sum of the external mass force  $\mathbf{f}_A^b$ , the force of interaction of the continua  $\mathbf{f}_A^{\text{int}}$  and the contact force

$$\mathbf{f}_A = \mathbf{f}_A^b + \mathbf{f}_A^{\text{int}} + \mathbf{f}_A^c = \int \rho_A \mathbf{g} dV + \int \mathbf{b}_A^{\text{int}} dV + \oint \mathbf{t}_A dS$$

where  $\mathbf{g}(\mathbf{x}, t)$  is the density of the external mass force,  $\mathbf{t}_A$  is the stress vector, and  $\mathbf{b}_A^{\text{int}}$  is the density of the force of interaction of the continuum  $A$  with the remaining components of the medium. Here and henceforth it is understood that the integration is carried out over the region  $\chi$  ( $dV$  is an element of volume) or over the surface  $\partial\chi$  ( $dS$  is an element of the surface).

The forces of interaction of the continua are equal in value and opposite in direction:

$$\mathbf{b}_S^{\text{int}} = -\mathbf{b}_F^{\text{int}} \quad (2.4)$$

Assumption (2.4) can be verified by averaging the laws of conservation of momentum for the components of the porous medium considered at the mesolevel.

It is assumed that the stress vector  $\mathbf{t}_A$  depends on the simplest characteristics of an element of the surface: the location  $\mathbf{x}$  and the orientation, which is specified by the outward normal  $\mathbf{n}$  of this element, i.e. the Cauchy postulate  $\mathbf{t}_A = \mathbf{t}_A(\mathbf{x}, \mathbf{n})$  holds.

The relation between the forces acting on the continuum  $A$  with the motion of this continuum is obtained by means of the laws of conservation of momentum and angular momentum about an arbitrary point  $\mathbf{x}_0$ , specified in an inertial frame of reference

$$\frac{d}{dt} \int \rho_A \mathbf{v}_A dV + \oint \mathbf{n} \cdot (\mathbf{v}_A \otimes \rho_A \mathbf{v}_A) dS = \int (\rho_A \mathbf{g} - \mathbf{b}_A^{\text{int}}) dV + \oint \mathbf{t}_A dS \quad (2.5)$$

$$\frac{d}{dt} \int \mathbf{r} \times \rho_A \mathbf{v}_A dV + \oint \mathbf{n} \cdot (\mathbf{v}_A \otimes \mathbf{r} \times \rho_A \mathbf{v}_A) dS = \int \mathbf{r} \times (\rho_A \mathbf{g} - \mathbf{b}_A^{\text{int}}) dV + \oint \mathbf{r} \times \mathbf{t}_A dS \quad (2.6)$$

where  $\mathbf{r} = \mathbf{x} - \mathbf{x}_0$ . Relations (2.5) and (2.6) hold if we can neglect the contribution of velocity pulsations to the acting force and moment of the force. Otherwise, pulsation and moment stresses occur in the conservation laws. Such a continuum is, in general, a micropolar medium.

For the stress vector, which obeys Cauchy's postulate, the law of conservation of momentum (2.5) enables us to prove the analogue of the fundamental Cauchy theorem<sup>14</sup>  $\mathbf{t}_A(\mathbf{x}, \mathbf{n}) = \boldsymbol{\sigma}_A(\mathbf{x}) \cdot \mathbf{n}$ , where  $\boldsymbol{\sigma}_A$  is the Cauchy partial stress tensor for the continuum  $A$ . A consequence of the law of conservation of angular momentum (2.6) is the symmetry  $\boldsymbol{\sigma}_A = \boldsymbol{\sigma}_A^T$  of the Cauchy partial stress tensors.

Taking into account the continuity equation (2.2) for smooth motions, from integral equation (2.5) we obtain the equation of motion of continuum A

$$\rho_A \frac{d_A \mathbf{v}_A}{dt} - \nabla \cdot \boldsymbol{\sigma}_A = \rho_A \mathbf{g} - \mathbf{b}_A^{\text{int}} \tag{2.7}$$

The interaction force, the partial stress and the porosity of the medium can be represented in the form of the sum of two terms

$$\mathbf{b}_A^{\text{int}} = \mathbf{b}_A^0 + \mathbf{b}_A^{\text{dis}}, \quad \boldsymbol{\sigma}_A = \boldsymbol{\sigma}_A^0 + \boldsymbol{\sigma}_A^{\text{dis}}, \quad \phi = \phi^0 + \phi^{\text{dis}} \tag{2.8}$$

where  $\mathbf{b}_A^0$ ,  $\boldsymbol{\sigma}_A^0$  and  $\phi^0$  are the equilibrium interaction force, stress and porosity, the  $\mathbf{b}_A^{\text{dis}}$ ,  $\boldsymbol{\sigma}_A^{\text{dis}}$  and  $\phi^{\text{dis}}$  are the dissipative interaction force, stress and porosity respectively. The dissipative terms are equal to zero in the equilibrium state, in which the temperature gradient and the mass velocities are equal to zero in a certain neighbourhood of the actual instant of time.

The equilibrium interaction force of the fluid with the skeleton is written in the form

$$\mathbf{b}_F^0 = \boldsymbol{\sigma}_f^0 \cdot \nabla \phi^0 \tag{2.9}$$

In fact, for the equilibrium state of the fluid, Eq. (2.7) reduces to the equilibrium equation  $\nabla \cdot \boldsymbol{\sigma}_F^0 = \rho_F^0 \mathbf{g} + \mathbf{b}_F^0$ . Using the relation between the average and true values (indicated by a lower-case letter)  $\boldsymbol{\sigma}_F^0 = \phi^0 \boldsymbol{\sigma}_f^0$ ,  $\rho_F^0 = \phi^0 \rho_f$ , the equilibrium equation can be written in the form  $-\phi^0 \nabla \cdot \boldsymbol{\sigma}_f^0 - \boldsymbol{\sigma}_f^0 \cdot \nabla \phi^0 = \phi^0 \rho_f \mathbf{g} - \mathbf{b}_F^0$

In the state of rest  $\nabla \cdot \boldsymbol{\sigma}_f^0 + \rho_f \mathbf{g} = 0$ . Hence formula (2.9) follows.

The representation of the interaction force in the form of the sum (2.8) is well known<sup>3,5</sup> for an ideal fluid. The main new factor is that expression (2.9) is treated as a universal relation, which holds for any fluid, unlike the dissipative term  $\mathbf{b}_F^{\text{dis}}$ , for which a governing relation of the Darcy-law type is formulated.

When describing a thermodynamically saturated porous medium, we will confine ourselves to the case when an element of the porous medium is characterized by a single absolute temperature  $\theta(\mathbf{x}, t) > 0$ . Of course, this approximation does not always hold. A small value of the velocity of relative motion in conjunction with a high thermal conductivity of the components is a non-rigorous condition for this approximation. When this condition, which depends on the properties of the material, the characteristic time and the scale of the process, is violated, it is necessary to introduce two temperatures.<sup>5</sup>

The first law of thermodynamics, which only takes into account thermal and mechanical interaction, is postulated in the form of an energy-balance axiom

$$\frac{d}{dt} \int \sum \rho_A e_A dV + \oint \mathbf{n} \cdot \sum \rho_A e_A \mathbf{v}_A dS = \int \sum \rho_A \mathbf{g} \cdot \mathbf{v}_A dV + \oint \sum \mathbf{t}_A \cdot \mathbf{v}_A dS + \int (\nabla \cdot \mathbf{q} + \rho r) dV \tag{2.10}$$

where  $u_A$  is the specific internal energy of continuum A,  $e_A = u_A + \mathbf{v}_A \cdot \mathbf{v}_A / 2$  is the specific total energy,  $\mathbf{q}$  is the heat flux,  $r$  is the volume heat source density and  $\sum \Phi_A = \Phi_S + \Phi_F$ . This form of writing the total energy does not take into account the effect of the ‘‘tortuosity’’ of the channels of the porous space. For small velocities of motion of the fluid, the contribution of pulsations to the kinetic energy can obviously be neglected. The energy balance axiom (2.10) also does not take into account surface (capillary) effects.<sup>3–5</sup>

For smooth motions, the integral energy balance (2.10), taking into account continuity equation (2.2) and equation of motion (2.7), can be written in the form of a local energy equation

$$\sum \rho_A \frac{d_A u_A}{dt} = \sum \boldsymbol{\sigma}_A : (\nabla \otimes \mathbf{v}_A) + \mathbf{b}^{\text{int}} \cdot \mathbf{w} + \nabla \cdot \mathbf{q} + \rho r \tag{2.11}$$

Here  $\mathbf{w} = \mathbf{v}_F - \mathbf{v}_S$  is the relative velocity and  $\mathbf{b}^{\text{int}} \equiv \mathbf{b}_F^{\text{int}}$  is the interaction force. It can be seen that the rate of variation of the internal energy depends on the strength of the stresses, the rate of heat supply and the strength of the interaction force.



The second law of thermodynamics for thermomechanical interaction of the components can be written in the form of an integral inequality

$$\frac{d}{dt} \int \rho_A \eta_A dV + \oint \mathbf{n} \cdot \sum \rho_A \eta_A \mathbf{v}_A dS \geq \int \frac{\rho r}{\theta} dV + \oint \frac{1}{\theta} \mathbf{n} \cdot \mathbf{q} dS \quad (2.12)$$

where  $\eta_A$  is the specific entropy of continuum  $A$ . Unlike classical thermodynamics, which only considers equilibrium states, no such limitations exist here. The state of a particle may vary in space and time. The requirement that local thermodynamic equilibrium of an element should exist imposes a limitation on the rate of change of the state – this rate should not be so large as to be able to introduce a temperature, stresses and other average parameters of state.

For smooth motions, from integral relations (2.12) we obtain the following local inequality for the rate of entropy production

$$\sum \rho_A \frac{d_A \eta_A}{dt} - \nabla \cdot (\mathbf{q}/\theta) - \rho r/\theta \geq 0, \quad \text{or} \quad \delta_M + \delta_T \geq 0 \quad (2.13)$$

where

$$\delta_M = \theta \frac{\partial}{\partial t} \sum \rho_A \eta_A + \theta \nabla \cdot \sum \rho_A \eta_A \mathbf{v}_A - \nabla \cdot \mathbf{q} - \rho r, \quad \delta_T = \frac{1}{\theta} \mathbf{q} \cdot \nabla \theta$$

$\delta_M$  and  $\delta_T$  are the internal and thermal dissipations respectively. In the majority of models it is assumed that each dissipation is negative. This assumption is sufficient to satisfy inequality (2.13), but is not entirely necessary.

Although in not a single problem, as a rule, is a formal check of the inequality for the rate of entropy production required, its value lies in the fact that considerable limitations, imposed on the governing relations follow from it.<sup>14</sup> To derive these limitations it is convenient to use the inequality in the form

$$-\sum \left( \rho_A \frac{d_A \psi_A}{dt} + \rho_A \eta_A \frac{d_A \theta}{dt} \right) - (\boldsymbol{\sigma} \cdot \mathbf{G}_S^{-1}) : \frac{d_S \mathbf{G}_S^t}{dt} + \boldsymbol{\sigma}_f^0 : (\nabla \otimes (\phi^0 \mathbf{w})) + \delta_f + \delta_T \geq 0 \quad (2.14)$$

where  $\delta_f = \sigma_F^{\text{dis}} : (\nabla \otimes \mathbf{w}) + \mathbf{b}^{\text{dis}} \cdot \mathbf{w}$  is the dissipation due to viscous interaction of the fluid with the skeleton (seepage dissipation), and  $\boldsymbol{\sigma}_F^0$  is the equilibrium tensor of the true stresses of the fluid, which occurs in the representation  $\boldsymbol{\sigma}_F = \phi_0 \boldsymbol{\sigma}_F^0 + \boldsymbol{\sigma}_F^{\text{dis}}$ .

To derive inequality (2.14) we can use expression (2.9) for the equilibrium interaction force and relation (1.4). We find

$$\sum \boldsymbol{\sigma}_A : (\nabla \otimes \mathbf{v}_A) + \mathbf{b}^{\text{int}} \cdot \mathbf{w} = -(\boldsymbol{\sigma} \cdot \mathbf{G}_S^{-1}) : d_S \mathbf{G}_S^T/dt + \boldsymbol{\sigma}_f^0 : (\nabla \otimes (\phi^0 \mathbf{w})) + \delta_f$$

Using this relation and the local energy balance equation (2.11), we arrive at the following expression for the rate of contact and bulk heating

$$\nabla \cdot \mathbf{q} + \rho r = \sum \rho_A d_A u_A/dt + (\boldsymbol{\sigma} \cdot \mathbf{G}_S^{-1}) : d_S \mathbf{G}_S^T/dt + \boldsymbol{\sigma}_f^0 : (\nabla \otimes (\phi^0 \mathbf{w})) - \delta_f$$

the substitution of which into relation (2.13) and the use of the expression for the free energy density  $\psi_A = u_A - \theta \eta_A$  gives inequality (2.14).

### 3. A thermoelastic saturated porous medium with viscous interaction of the components

The state of the element of such a medium at the point  $(\mathbf{x}, t)$  will be specified by the quantities

$$\boldsymbol{\Lambda}(\mathbf{x}, t) = \{ \mathbf{G}_S, \mathbf{G}_F, \theta, \boldsymbol{\gamma}, \mathbf{w} \} \quad (3.1)$$

where  $\mathbf{G}_A$  are the distortions,  $\mathbf{w}$  is the relative velocity of the fluid and  $\boldsymbol{\gamma} = \nabla \theta$  is the temperature gradient. The reaction of the material is characterized by the quantities

$$\boldsymbol{\Upsilon}(\mathbf{x}, t) = \{ \psi_A(\mathbf{x}, t), \boldsymbol{\sigma}_A(\mathbf{x}, t), \eta_A(\mathbf{x}, t), \phi(\mathbf{x}, t), \mathbf{q}(\mathbf{x}, t), \mathbf{b}^{\text{dis}}(\mathbf{x}, t) \}$$

These quantities are assumed to be functions of the actual state of the skeleton and the fluid

$$\Upsilon(\mathbf{x}, t) = \Upsilon^+ \{ \Lambda(\mathbf{x}, t) \} \tag{3.2}$$

where  $\Upsilon^+ = \{ \psi_A^+, \boldsymbol{\sigma}_A^+, \eta_A^+, \phi^+, \mathbf{q}^+, \mathbf{b}^+ \}$  is a set of defining functions.

The superscript plus (or other similar symbols) is used here and henceforth in order to distinguish functions which depend on different arguments but have the same value.

As an example we have the free energy of the skeleton

$$\psi_S = \psi_S^+(\mathbf{V}_S, \dots) = \psi_S^x(\mathbf{E}_S, \dots), \quad \mathbf{E}_S = (\mathbf{V}_S^{-2} - \mathbf{I})/2$$

expressed in the first case in terms of the symmetrical tensor  $\mathbf{V}_S$ , which occurs in polar expansion (1.2), and in the second case in terms of the Lagrange finite deformation tensor.

It is assumed that, in the initial state, the fluid and the skeleton have constant temperature and stress. The material is assumed to be uniform, and hence there are no Lagrange radius vectors  $\mathbf{X}_A$  in a number of the parameters of state. It follows from the fact that the governing relations are independent of the choice of the reference frame that the reaction depends on the relative velocity  $\mathbf{w}$ , and not on the mass velocity  $\mathbf{v}_A$ .

There are several features in the definition of the material (3.1), (3.2). First, the total interaction force does not occur in system (3.2), but only its dissipative component  $\mathbf{b}^{\text{dis}}$ , since the equilibrium component is given by expression (2.9), irrespective of the fluid properties. Moreover, it is assumed that the reaction of the continuum A depends on the distortions of the skeleton and fluid. The temperature gradient and the relative velocity, being “natural” for the heat flux and interaction force, by virtue of the equipresence rule are also included in the number of arguments of all the defining functions.

Unlike a classical thermoelastic medium, the relative velocity occurs in a number of state parameters, which, strictly speaking, remove the medium considered from the class of materials not possessing a memory of former states. However, as will be seen later, a dependence on the velocity is only characteristic of the heat flux and the interaction force. The thermodynamic potentials and the stresses and entropy defined by them do not depend on the velocity.

Unlike a single-component material, a saturated porous body is made up of two continua with their own initial states. Hence, the problem arises of transformations of the reference configurations, which leave the governing equations unchanged. The gradients of these transformations form a symmetry group of the saturated porous medium, representing a set of two groups:  $g = \{g_{\kappa_S}, g_{\kappa_F}\}$ , which, in general, depend on the chosen configurations  $\kappa_A$ ; the change in these groups on transferring to another initial configuration is described by the well known Noll rule.<sup>14</sup>

It is assumed that the porous medium is saturated with a thermoelastic fluid, while the skeleton is a thermoelastic solid and an undistorted configuration  $\kappa_0$  exists for it, the symmetry group in which  $g_0 \in o$  is a subgroup of the inherent orthogonal group. Then, the governing equations possess the symmetry group  $g = \{g_0, u\}$  for any reference configuration of the fluid  $\kappa_F$ . Here  $u$  is the characteristic unimodular group of second-rank tensors, the determinant of which is equal to +1.

Another general postulate of the theory of governing equations – the principle of independence of the choice of the reference frame<sup>13,14</sup> (or the principle of material objectivity) asserts that governing relations (3.2) must be invariant under a change in the reference frame, specified by the transformation

$$\mathbf{x}^* = \mathbf{Q}(t) \cdot (\mathbf{x} - \mathbf{x}_0) + \mathbf{x}_1(t), \quad t^* = t - a, \quad \mathbf{x}_1(-\infty) = \mathbf{x}_0, \quad \mathbf{Q}(-\infty) = \mathbf{I} \tag{3.3}$$

where  $\mathbf{Q}(t)$  is the orthogonal tensor of rotation of the new reference frame with respect to the old one,  $\mathbf{x}_1(t) - \mathbf{x}_0$  is the parallel-transfer vector,  $\mathbf{x}_0 = \text{const}$  is the point, relative to which the rotation occurs, and  $a = \text{const}$  is the value of the shift in time. For simplicity it is assumed that the reference frames coincide as  $t \rightarrow -\infty$ .

The invariance of the defining mappings (3.2) with respect to transformations (3.3) denotes that the relation of the state  $\Lambda^* = (\mathbf{G}_S^*, \mathbf{G}_F^*, \boldsymbol{\theta}^*, \boldsymbol{\gamma}^*, \mathbf{w}^*)$  to the actual reaction  $\Upsilon^*$  is accomplished by the same function, i.e.  $\Upsilon^* = \Upsilon^+(\Lambda^*)$ .

Taking into account the fact that when there is a change in the reference frame  $d\mathbf{x}^* = \mathbf{Q}(t) \cdot d\mathbf{x}$ ,  $d\mathbf{X}_A^* = d\mathbf{X}_A$ , from the definitions of the distortion, the temperature gradient, the relative velocity and the uniqueness of the polar



expansions (1.2) we obtain

$$\begin{aligned} \mathbf{G}_A^* &= \mathbf{G}_A \cdot \mathbf{Q}^T, \quad \mathbf{U}_A^* = \mathbf{Q} \cdot \mathbf{U}_A \cdot \mathbf{Q}^T, \quad \mathbf{V}_A^* = \mathbf{V}_A, \quad \mathbf{R}_A^* = \mathbf{R}_A \cdot \mathbf{Q}^T, \\ \boldsymbol{\gamma}^* &= \mathbf{Q} \cdot \boldsymbol{\gamma}, \quad \mathbf{w}^* = \mathbf{Q} \cdot \mathbf{w} \end{aligned} \quad (3.4)$$

Assuming that the free energy, entropy, temperature and porosity remain unchanged when the reference frame is changed, the heat flux and the interaction force are transformed like vectors, while the Cauchy partial-stress tensors are “objective” tensors, we have

$$\begin{aligned} \psi_A^* &= \psi_A, \quad \eta_A^* = \eta_A, \quad \theta^* = \theta, \quad \phi^* = \phi, \quad \mathbf{q}^* = \mathbf{Q} \cdot \mathbf{q}, \quad \mathbf{b}^{\text{int}*} = \mathbf{Q} \cdot \mathbf{b}^{\text{int}}, \\ \boldsymbol{\sigma}_A^* &= \mathbf{Q} \cdot \boldsymbol{\sigma}_A \cdot \mathbf{Q}^T \end{aligned} \quad (3.5)$$

The principle of thermodynamic consistency<sup>14</sup> implies that inequality (2.14) for the rate of entropy production holds for any history of the states. In combination with the requirement that the governing relations of the porous medium should be independent of the choice of the reference frame, it leads to extremely strong limitations. To satisfy these principles it is necessary and sufficient that the governing equations of a thermoelastic saturated porous medium with any type of symmetry of the compressible skeleton should have the form

$$\begin{aligned} \psi_F &= \psi_F(\rho_f, \theta), \quad \boldsymbol{\sigma}_F = -\phi p_f \mathbf{I}, \quad p_f = \rho_f^2 \frac{\partial \psi_F}{\partial \rho_f}, \quad \eta_F = -\frac{\partial \psi_F}{\partial \theta} \\ \psi_S &= \psi_\phi(\mathbf{V}_S, \phi, \theta), \quad \boldsymbol{\sigma}_S = -\rho_S \frac{\partial \psi_\phi}{\partial \mathbf{G}_S^T} \cdot \mathbf{G}_S, \quad p_f = \rho_S \frac{\partial \psi_\phi}{\partial \phi}, \quad \eta_S = -\frac{\partial \psi_\phi}{\partial \theta} \end{aligned} \quad (3.6)$$

$$\Pi(\mathbf{V}_S, \phi, \theta, \rho_f) \equiv \rho_f^2 \frac{\partial \psi_F(\rho_f, \theta)}{\partial \rho_f} - \rho_S \frac{\partial \psi_\phi(\mathbf{V}_S, \phi, \theta)}{\partial \phi} = 0 \quad (3.7)$$

$$\begin{aligned} \mathbf{q} &= \mathbf{R}_S^T \cdot \mathbf{q}^\times(\mathbf{V}_S, \phi, \theta, \boldsymbol{\gamma}^R, \mathbf{w}^R), \quad \mathbf{b}^{\text{dis}} = \mathbf{R}_S^T \cdot \mathbf{b}^\times(\mathbf{V}_S, \phi, \theta, \boldsymbol{\gamma}^R, \mathbf{w}^R) \\ \delta_f + \delta_T &\geq 0, \quad \delta_f = \mathbf{b}^\times \cdot \mathbf{w}^R, \quad \delta_T = \theta^{-1} \mathbf{q}^\times \cdot \boldsymbol{\gamma}^R, \quad \boldsymbol{\gamma}^R = \mathbf{R}_S \cdot \boldsymbol{\gamma}, \quad \mathbf{w}^R = \mathbf{R}_S \cdot \mathbf{w} \end{aligned} \quad (3.8)$$

Here and henceforth  $\rho_f$  is the true density of the fluid,  $p_f$  is the pore pressure, and  $\mathbf{R}_S$  and  $\mathbf{V}_S$  are orthogonal and symmetrical tensors, which occur in expansion (1.2).

Conditions (3.6) indicate that the free energy of the fluid is independent of the distortion of the skeleton, the temperature gradient and the relative velocity, and is a function of the fluid density and the temperature of the medium. The fluid stress tensor is a spherical tensor, so that the deviator of the equilibrium stress and the dissipative stress of the fluid are identically equal to zero. The pore pressure and the entropy of the fluid are determined by the partial derivatives of the free energy and, consequently, are also independent of the temperature gradient and the relative velocity.

The porosity is a function of the temperature, the deformation of the skeleton and the fluid density and is identical with the porosity in the equilibrium state. This function is defined by the equation  $\Pi(\mathbf{V}_S, \phi, \theta, \rho_f) = 0$ , specified by the potentials of the components. The stress tensor and the entropy of the skeleton are determined by the free energy of the skeleton  $\psi_\phi(\mathbf{V}_S, \phi, \theta)$ , which is independent of the temperature gradient and the relative velocity.

Formulae (3.7) give the structure of the expressions for the heat flux and the dissipative interaction force. In general, the heat flux and the dissipative interaction force depend both on the temperature gradient and on the relative velocity, i.e. the heat and mass transfer are related.

Relations (3.8) show that, for any state of the material considered, the internal dissipation is identical with the seepage dissipation, and the sum of the dissipations is non-negative, but, in general, nothing can be said about the sign of each of the components.

The relations obtained hold for arbitrary deformations and changes in the temperature of the medium for any type of anisotropy of the skeleton.

Proof of relations (3.6)–(3.8). We will fix the point  $(\mathbf{x}_0, t_0)$ . We will take as an element of the symmetry group  $g_{\kappa_F} = u$  the unimodular tensor  $P = J_F^{1/3} \mathbf{G}_F^{-1}(\mathbf{x}_0, t_0)$ . With this transformation of the reference configuration  $\kappa_F$  (and

unchanged configuration  $\kappa_S$ ) the distortion  $\mathbf{G}_F \rightarrow \mathbf{P}_F \cdot \mathbf{G}_F = J_F^{1/3} \mathbf{I}$ , while the distortion  $\mathbf{G}_S$  and the vectors  $\boldsymbol{\gamma}, \mathbf{w}$  remain unchanged. By virtue of the symmetry of the material, the functions  $\Upsilon^+(\Lambda)$  occurring in governing relations (3.2) remain invariant, i.e.

$$\Upsilon^+(\mathbf{G}_S, \mathbf{G}_F, \boldsymbol{\theta}, \boldsymbol{\gamma}, \mathbf{w}) = \Upsilon^+(\mathbf{G}_S, \mathbf{P} \cdot \mathbf{G}_F, \boldsymbol{\theta}, \boldsymbol{\gamma}, \mathbf{w}) = \Upsilon^+(\mathbf{G}_S, J_F^{1/3} \mathbf{I}, \boldsymbol{\theta}, \boldsymbol{\gamma}, \mathbf{w}) = \Upsilon^+(\mathbf{G}_S, \rho_f, \boldsymbol{\theta}, \boldsymbol{\gamma}, \mathbf{w})$$

In the last equality we have taken into account the law of conservation of mass (2.1) and the presence of a governing equation for the porosity, which enables the average fluid density to be expressed in terms of its true density.

We will apply to these equations the principle that the governing relations are independent of a change of the reference frame. We will choose the reference frame so that it is turned together with the neighbourhood of the point on the skeleton considered. Then, the tensor  $\mathbf{Q}(t)$  in relation (3.3) is equal to the orthogonal tensor  $\mathbf{R}_S(\mathbf{x}_0, t)$  of polar expansion (1.2). This replacement of the reference frame, taking formulae (3.4) and (3.5) into account, gives

$$\Lambda^* = \{\mathbf{G}_S^*, \rho_f^*, \boldsymbol{\theta}^*, \boldsymbol{\gamma}^*, \mathbf{w}^*\} = \{\mathbf{V}_S, \rho_f, \boldsymbol{\theta}, \boldsymbol{\gamma}^R, \mathbf{w}^R\}, \quad \boldsymbol{\gamma}^R = \mathbf{R}_S \cdot \boldsymbol{\gamma}, \quad \mathbf{w}^R = \mathbf{R}_S \cdot \mathbf{w}$$

$$\Upsilon^* = \{\Psi_A^*, \boldsymbol{\sigma}_A^*, \eta_A^*, \phi^*, \mathbf{q}^*, \mathbf{b}^{\text{dis}*}\} = \{\Psi_A, \mathbf{Q} \cdot \boldsymbol{\sigma}_A \cdot \mathbf{Q}^T, \eta_A, \phi, \mathbf{Q} \cdot \mathbf{q}, \mathbf{Q} \cdot \mathbf{b}^{\text{dis}}\}$$

From the property of objectivity  $\Upsilon^* = \Upsilon^\times\{\Lambda^*\}$ , where the functions  $\Upsilon^\times$  are the same in the new and old reference frame, we arrive at the need for the relations

$$\begin{aligned} \Psi_A &= \Psi_A^\times(\Lambda^R), \quad \boldsymbol{\sigma}_A = \mathbf{R}_S^T \cdot \boldsymbol{\sigma}_A^\times(\Lambda^R) \cdot \mathbf{R}_S, \quad \eta_A = \eta_A^\times(\Lambda^R), \quad \phi = \phi^\times(\Lambda^R) \\ \mathbf{q} &= \mathbf{R}_S^T \cdot \mathbf{q}^\times(\Lambda^R), \quad \mathbf{b}^{\text{dis}} = \mathbf{R}_S^T \cdot \mathbf{b}^\times(\Lambda^R), \quad \Lambda^R \equiv (\mathbf{V}_S, \rho_f, \boldsymbol{\theta}, \boldsymbol{\gamma}^R, \mathbf{w}^R) \end{aligned} \tag{3.9}$$

The sufficiency of relations (3.9) for invariance for arbitrary unimodular transformations of any reference frame  $\kappa_F$  of the fluid and arbitrary orthogonal transformations of the actual configuration of the body is obvious, if we take formulae (3.4) and (3.5) into account.

We now turn to inequality (2.14). Note that, for a medium with a compressible skeleton, saturated with a compressible fluid, the divergence of the seepage rate  $\phi \mathbf{w}$  can be expressed in terms of the derivatives of the distortion of the skeleton, the fluid density and the porosity

$$\phi \rho_f^{-1} \frac{d_F \rho_f}{dt} + \frac{d_S \phi}{dt} - \phi \mathbf{G}_S^{-1} : \frac{d_S \mathbf{G}_S^T}{dt} + \nabla \cdot (\phi \mathbf{w}) = 0 \tag{3.10}$$

It can be shown by a direct check that this relation is equivalent to Eq. (2.3). Since relation (3.10) is satisfied for any state, it also holds for the history of equilibrium states. Then

$$\boldsymbol{\sigma}_f^0 : (\nabla \otimes (\phi^0 \mathbf{w})) = \mathbf{s}_f^0 : (\nabla \otimes (\phi^0 \mathbf{w})) - p_f^0 \phi^0 \mathbf{G}_S^{-1} : \frac{d_S \mathbf{G}_S^T}{dt} + p_f^0 \frac{d_S \phi^0}{dt} + \frac{p_f^0 \phi^0}{\rho_f} \frac{d_F \rho_f}{dt}$$

where  $\mathbf{s}_f^0$  is deviator of the equilibrium stress in the fluid. Inequality (2.14) for the case considered can be converted to the form

$$\begin{aligned} & - \sum \rho_A \left( \frac{d_A \Psi_A}{dt} + \eta_A \frac{d_A \boldsymbol{\theta}}{dt} \right) - (\boldsymbol{\sigma}^{\text{eq}} \cdot \mathbf{G}_S^{-1}) : \frac{d_S \mathbf{G}_S^T}{dt} + p_f^0 \frac{d_S \phi^0}{dt} + \frac{\phi^0 p_f^0}{\rho_f} \frac{d_F \rho_f}{dt} + \\ & + \mathbf{s}_f^0 : (\nabla \otimes (\phi^0 \mathbf{w})) + \delta_f + \delta_T \geq 0 \end{aligned}$$

where  $\boldsymbol{\sigma}^{\text{eq}} = \boldsymbol{\sigma} + p_f^0 \phi^0 \mathbf{I} = \boldsymbol{\sigma} + p_F^0 \mathbf{I}$  is the equivalent stress.

Suppose  $\Lambda_0 = \{\mathbf{G}_S, \rho_f, \boldsymbol{\theta}, \boldsymbol{\gamma}, \mathbf{w}\}$  is an arbitrary state of the saturated porous medium at the point  $\mathbf{x}_0$  at the instant of time  $t_0$ . We will specify the state in a small neighbourhood of this point in the form of the linear function

$$\Lambda(\mathbf{x}_0, t) = \Lambda_0 + a(t - t_0) + (\mathbf{x} - \mathbf{x}_0) \cdot \mathbf{B}.$$

where  $\mathbf{a}$  and  $\mathbf{B}$  are the rates of change and the gradients of the state parameters at the point  $\mathbf{x}_0$ . These quantities, apart from the temperature gradient, can take arbitrary values, which are independent of the state  $\Lambda_0$ . The derivatives  $d_A/dt$  of the state parameters and the tensor  $\nabla \otimes (\phi^0 \mathbf{w})$  are then also arbitrary.

By virtue of the first formula of (3.9) the functions of the free energy  $\psi_A$  and of the porosity  $\phi^0$  in the equilibrium state can be written in the form of the relations

$$\psi_A = \psi_A(\mathbf{G}_S, \rho_f, \theta, \boldsymbol{\gamma}, \mathbf{w}), \quad \phi^0 = \phi^0(\mathbf{G}_S, \rho_f, \theta)$$

Calculating their derivatives with respect to time and substituting the expressions obtained into the inequality for the rate of entropy production, we obtain when  $t \geq t_0$

$$\begin{aligned} & - \left( \rho_S \frac{\partial \psi_S}{\partial \mathbf{G}_S^T} + \boldsymbol{\sigma}^{\text{eq}} \cdot \mathbf{G}_S^{-1} - p_f^0 \frac{\partial \phi^0}{\partial \mathbf{G}_S^T} \right) : \frac{d_S \mathbf{G}_S^T}{dt} - \rho_F \frac{\partial \psi_F}{\partial \mathbf{G}_S^T} : \frac{d_F \mathbf{G}_S^T}{dt} - \sum \rho_A \frac{\partial \psi_A}{\partial \mathbf{w}} \cdot \frac{d_A \mathbf{w}}{dt} - \\ & - \left( \rho_S \frac{\partial \psi_S}{\partial \rho_f} - p_f^0 \frac{\partial \phi^0}{\partial \rho_f} \right) \frac{d_S \rho_f}{dt} - \left( \rho_F \frac{\partial \psi_F}{\partial \rho_f} - \frac{p_f^0 \phi^0}{\rho_f} \right) \frac{d_F \rho_f}{dt} - \sum \rho_A \frac{\partial \psi_A}{\partial \boldsymbol{\gamma}} \cdot \frac{d_A \boldsymbol{\gamma}}{dt} - \\ & - \left( \rho_S \frac{\partial \psi_S}{\partial \theta} + \rho_S h_S - p_f^0 \frac{\partial \phi^0}{\partial \theta} \right) \frac{d_S \theta}{dt} - \rho_F \left( \frac{\partial \psi_F}{\partial \theta} + \eta_F \right) \frac{d_F \theta}{dt} + \\ & + (\boldsymbol{\sigma}_F^{\text{dis}} \cdot \mathbf{G}_S^{-1}) : \frac{d_S \mathbf{G}_S^T}{dt} - (\boldsymbol{\sigma}_F^{\text{dis}} \cdot \mathbf{G}_F^{-1}) : \frac{d_F \mathbf{G}_F^T}{dt} + \mathbf{s}_f^0 : (\nabla \otimes (\phi^0 \mathbf{w})) - \mathbf{b}^{\text{dis}} \cdot \mathbf{w} + \delta_T \geq 0 \end{aligned}$$

The coefficients of the derivatives and the quantity  $(\delta_T + \mathbf{b}^{\text{dis}} \cdot \mathbf{w})$  are functions of the state  $\Lambda_0$  by virtue of definition (3.2) of the material considered. These functions are independent of the derivatives with respect to time and the gradients of the state parameters. If the reaction  $\Upsilon^+(\Lambda(\mathbf{x}, t))$ , together with the partial derivatives of the functions  $\psi_A$  and  $\phi^0$  with respect to the state parameters is continuous at the point  $\Lambda_0$ , then, letting  $\mathbf{x} \rightarrow \mathbf{x}_0$  and  $t \rightarrow t_0$ , we obtain, by virtue of the fact that the derivatives and gradients of the state parameters occurring in the inequality are independent,

$$\frac{\partial \psi_A}{\partial \mathbf{w}} = 0, \quad \frac{\partial \psi_A}{\partial \boldsymbol{\gamma}} = 0, \quad \mathbf{s}_f^0 = 0, \quad \boldsymbol{\sigma}_F^{\text{dis}} = 0, \quad \frac{\partial \psi_F}{\partial \mathbf{G}_S} = 0 \quad (3.11)$$

$$\rho_F \frac{\partial \psi_F}{\partial \rho_f} = \frac{p_f^0 \phi^0}{\rho_f} \quad (3.12)$$

$$\rho_S \frac{\partial \psi_S}{\partial \mathbf{G}_S^T} + \boldsymbol{\sigma}^{\text{eq}} \cdot \mathbf{G}_S^{-1} - p_f^0 \frac{\partial \phi^0}{\partial \mathbf{G}_S^T} = 0, \quad \rho_S \frac{\partial \psi_S}{\partial \rho_f} - p_f^0 \frac{\partial \phi^0}{\partial \rho_f} = 0 \quad (3.13)$$

$$\rho_S \frac{\partial \psi_S}{\partial \theta} - p_f^0 \frac{\partial \phi^0}{\partial \theta} + \rho_S \eta_S = 0, \quad \frac{\partial \psi_F}{\partial \theta} + \eta_F = 0 \quad (3.14)$$

$$\delta_f + \delta_T \geq 0, \quad \delta_f = -\mathbf{b}^{\text{dis}} \cdot \mathbf{w}, \quad \delta_T = \boldsymbol{\theta}^{-1} \mathbf{q} \cdot \boldsymbol{\gamma}$$

Relations (3.11) show that the free energy of the components is independent of the temperature gradient and the velocity of relative motion. The deviator of the equilibrium stress and the dissipative-stress tensor in the fluid are equal to zero, by virtue of which the stress tensor is spherical, i.e.  $\boldsymbol{\sigma}_F = -p_F^0 \mathbf{I}$ . The free energy of the fluid is independent of the distortion of the skeleton, and hence  $\psi_F = \psi_F(\rho_f, \theta)$ .

Relation (3.12) can be written in the form

$$\phi \rho_f^2 \frac{\partial \psi_F^+(\rho_f, \theta)}{\partial \rho_f} = p_f^0(\mathbf{G}_S, \rho_f, \theta) \phi^0(\mathbf{G}_S, \rho_f, \theta)$$

Hence it follows that the porosity of a thermoelastic material  $\phi = \phi(\mathbf{G}_S, \rho_f, \theta)$ . This means that the difference between the porosity  $\phi^0$  in the equilibrium state and the porosity  $\phi$  vanishes. Taking the equality  $\boldsymbol{\sigma}_F^{\text{dis}} = 0$  into account, relation (3.12) leads to the following expression for the pore pressure of the fluid

$$p_f^0 = p_f = \rho_f^2 \frac{\partial \Psi_F}{\partial \rho_f}$$

The equivalent stress in the medium considered is identical with the partial stress of the skeleton

If  $\partial \phi / \partial \rho_f \neq 0$ , i.e. there is a one-to-one correspondence between the porosity and the true density of the fluid, we can transfer from the arguments  $(\mathbf{G}_S, \rho_f, \theta)$  of the potential  $\psi_S$  to the variable  $(\mathbf{G}_S, \phi, \theta)$ . Putting

$$\psi_S(\mathbf{G}_S, \rho_f, \theta) = \psi_\phi(\mathbf{G}_S, \phi(\mathbf{G}_S, \rho_f, \theta), \theta)$$

and taking into account the rule for the differentiation of a complex function, we can write relations (3.13) in the form

$$\boldsymbol{\sigma}_S \cdot \mathbf{G}_S^{-1} = -\rho_S \frac{\partial \psi_\phi(\mathbf{G}_S, \phi, \theta)}{\partial \mathbf{G}_S^T}, \quad p_f(\rho_f, \theta) = \rho_S \frac{\partial \psi_\phi(\mathbf{G}_S, \phi, \theta)}{\partial \phi}$$

Since the pore pressure  $p_f = \rho_f^2 \partial \psi_F / \partial \rho_f$ , we arrive at an equation for the porosity

$$\Pi(\mathbf{V}_S, \phi, \theta, \rho_f) \equiv \rho_f^2 \frac{\partial \psi_F(\rho_f, \theta)}{\partial \rho_f} - \rho_S \frac{\partial \psi_\phi(\mathbf{V}_S, \phi, \theta)}{\partial \phi} = 0$$

It can be seen that the relation  $\phi(\mathbf{V}_S, \rho_f, \theta)$  is determined by the potentials of the components.

It follows from relations (3.14) that the entropy of the liquid and the skeleton are equal to the partial derivatives

$$\eta_F = -\frac{\partial \psi_F^+(\rho_f, \theta)}{\partial \theta}, \quad \eta_S = -\frac{\partial \psi_\phi(\mathbf{G}_S, \phi, \theta)}{\partial \theta}$$

where the entropy of the fluid is calculated by differentiation at constant true mass density, while the entropy of the skeleton is calculated by differentiation at constant porosity.

Hence, when relations (3.9) are taken into account, we arrive at the form (3.6)–(3.8) of the governing relations, necessary for objectivity and thermodynamic consistency.

Conversely, if relations (3.6)–(3.8) are satisfied for any state  $\Lambda(t)$ , this is sufficient to satisfy inequality (2.14) and the independence of the choice of the reference frame.

The form of Eqs. (3.6)–(3.8), necessary and sufficient to satisfy the principles of objectivity and thermodynamic consistency of the governing relations of a thermoelastic saturated porous medium with an arbitrary type of skeleton symmetry, is not unique. If we change from the variables  $(\mathbf{V}_S, \phi, \theta)$  to describe the state of the skeleton to the variables  $(\mathbf{V}_S, p_f, \theta)$ , the governing relations take the form<sup>1,4</sup>

$$\begin{aligned} \boldsymbol{\sigma} &= -\rho_S \frac{\partial \Psi(\mathbf{V}_S, p_f, \theta)}{\partial \mathbf{G}_S^T} \cdot \mathbf{G}_S, \quad \phi = -\rho_S \frac{\partial \Psi(\mathbf{V}_S, p_f, \theta)}{\partial p_f}, \quad \eta_S = -\frac{\partial \Psi(\mathbf{V}_S, p_f, \theta)}{\partial \theta} \\ \mathbf{q} &= \mathbf{R}_S^T \cdot \mathbf{q}^\times(\mathbf{V}_S, p_f, \theta, \boldsymbol{\gamma}^R, \mathbf{w}^R), \quad \mathbf{b}^{\text{dis}} = \mathbf{R}_S^T \cdot \mathbf{b}^\times(\mathbf{V}_S, p_f, \theta, \boldsymbol{\gamma}^R, \mathbf{w}^R), \quad \delta_f + \delta_T \geq 0 \end{aligned} \tag{3.15}$$

where the potential  $\Psi$  is the solution of the equation

$$\Psi - p_f \frac{\partial \Psi}{\partial p_f} = \psi_\phi(\mathbf{V}_S, \phi, \theta)$$

In fact, it follows from relations (3.6) that

$$d\psi_\phi(\mathbf{G}_S, \phi, \theta) = -\rho_S^{-1} (\boldsymbol{\sigma}_S \cdot \mathbf{G}_S^{-1}) : d\mathbf{G}_S + \rho_S^{-1} p_f d\phi - \eta_S d\theta$$

Adding to both sides of this equation the quantity

$$-d(\rho_S^{-1} \phi p_f) = -\rho_S^{-1} p_f d\phi - \rho_S^{-1} \phi dp_f + \rho_S^{-1} \phi p_f \mathbf{G}_S^{-1} : d\mathbf{G}_S$$

we obtain, taking the equality  $\boldsymbol{\sigma} = \boldsymbol{\sigma}_S - \phi p_f \mathbf{I}$  into account, the thermodynamic identity

$$d(\psi_\phi - \rho_S^{-1} \phi p_f) = -\rho_S^{-1} \boldsymbol{\sigma} \cdot \mathbf{G}_S^{-1} : d\mathbf{G}_S - \eta_S d\theta - \rho_S^{-1} \phi dp_f$$

from which relations (3.15) follow.

Using the Lagrange finite deformation tensor  $\mathbf{E}_S = (\mathbf{V}_S^{-2} - \mathbf{I})/2$ , which is more convenient for applications, and taking into account the formulae

$$2 \frac{\partial \mathbf{E}}{\partial \mathbf{G}^{-1}} = (\mathbf{I} \otimes \mathbf{G}^{-1})^{(1432)} + (\mathbf{I} \otimes \mathbf{G}^{-1})^{(2431)}, \quad \frac{\partial \mathbf{G}^{-1}}{\partial \mathbf{G}} = -(\mathbf{G}^{-1} \otimes \mathbf{G}^{-1})^{(1342)}$$

relations (3.15) can be written in the form

$$\begin{aligned} \Psi &= \Psi(\mathbf{E}_S, p_f, \theta), \quad \boldsymbol{\sigma} = \rho_S \mathbf{G}_S^{-1} \cdot \frac{\partial \Psi}{\partial \mathbf{E}_S} \cdot \mathbf{G}_S^{-1T}, \quad \phi = -\rho_S \frac{\partial \Psi}{\partial p_f}, \quad \eta_S = -\frac{\partial \Psi}{\partial \theta} \\ \mathbf{q} &= \mathbf{R}_S^T \cdot \mathbf{q}^\times(\mathbf{E}_S, p_f, \theta, \boldsymbol{\gamma}^R, \mathbf{w}^R), \quad \mathbf{b}^{\text{dis}} = \mathbf{R}_S^T \cdot \mathbf{b}^\times(\mathbf{E}_S, p_f, \theta, \boldsymbol{\gamma}^R, \mathbf{w}^R), \quad \delta_f + \delta_T \geq 0 \end{aligned}$$

If we introduce the scalar potential

$$\Phi(\boldsymbol{\pi}, p_f, \theta) = \Psi - \rho_{\kappa_S}^{-1} \boldsymbol{\pi} : \mathbf{E}_S, \quad \boldsymbol{\pi} = J_S \mathbf{G}_S \cdot \boldsymbol{\sigma} \cdot \mathbf{G}_S^T$$

where  $\boldsymbol{\pi}$  is a symmetrical tensor of the total Piola-Kirchhoff stresses of the second kind and  $\rho_{\kappa_S}$  is the density of the skeleton in the reference configuration  $\kappa_S$ , we arrive at a system of governing equations of the thermoelastic saturated porous medium in the variables  $(\boldsymbol{\pi}, p_f, \theta)$

$$\begin{aligned} \Phi &= \Phi(\boldsymbol{\pi}, p_f, \theta), \quad \mathbf{E}_S = -\rho_{\kappa_S} \frac{\partial \Phi}{\partial \boldsymbol{\pi}}, \quad \phi_\kappa = -\rho_{\kappa_S} \frac{\partial \Phi}{\partial p_f}, \quad \eta_S = -\frac{\partial \Phi}{\partial \theta} \\ \mathbf{q} &= \mathbf{R}_S^T \cdot \mathbf{q}^-(\boldsymbol{\pi}, p_f, \theta, \boldsymbol{\gamma}^R, \mathbf{w}^R), \quad \mathbf{b}^{\text{dis}} = \mathbf{R}_S^T \cdot \mathbf{b}^-(\boldsymbol{\pi}, p_f, \theta, \boldsymbol{\gamma}^R, \mathbf{w}^R), \quad \delta_f + \delta_T \geq 0 \end{aligned} \tag{3.16}$$

where  $\phi_\kappa = J_S \phi$  is the relative porosity. System of equations (3.16) possesses the simplest structure compared with systems (3.6) and (3.15), since the strain tensor and the relative porosity are equal to (apart from constant factors) the derivatives of the thermodynamic potential.

**Corollary 1.** *The heat flux is equal to zero in the equilibrium state, i.e.*

$$\mathbf{q}^\times(\mathbf{E}_S, p_f, \theta, 0, 0) = 0 \tag{3.17}$$

while the second-rank symmetrical tensors of the equilibrium thermal conductivity, resistance and mutual influence of the heat and mass fluxes

$$\mathbf{Y}_\gamma = \left. \frac{1}{\theta} \frac{\partial \mathbf{q}^\times}{\partial \boldsymbol{\gamma}^R} \right|_0, \quad \mathbf{Y}_w = -\left. \frac{\partial \mathbf{b}^\times}{\partial \mathbf{w}^R} \right|_0, \quad \mathbf{Y}_{\gamma w} = \text{sym} \left( \left. \frac{1}{\theta} \frac{\partial \mathbf{q}^\times}{\partial \mathbf{w}^R} - \frac{\partial \mathbf{b}^\times}{\partial \boldsymbol{\gamma}^R} \right) \right|_0$$

possess the following properties

$$\mathbf{Y}_\gamma : (\mathbf{a} \otimes \mathbf{a}) \geq 0, \quad \mathbf{Y}_w : (\mathbf{b} \otimes \mathbf{b}) \geq 0, \quad (\mathbf{Y}_\gamma : (\mathbf{a} \otimes \mathbf{a}))(\mathbf{Y}_w : (\mathbf{b} \otimes \mathbf{b})) - (\mathbf{Y}_{\gamma w} : (\mathbf{a} \otimes \mathbf{b}))^2 \geq 0 \tag{3.18}$$

for all  $\mathbf{a} \neq 0, \mathbf{b} \neq 0$ .

In fact, it follows from relations (3.8) that when  $\boldsymbol{\gamma} = \mathbf{w} = 0$ , the overall dissipation

$$\delta = \boldsymbol{\theta}^{-1} \boldsymbol{\gamma}^R \cdot \mathbf{q}^\times + \mathbf{w}^R \cdot \mathbf{b}^\times = 0$$

This indicates that for fixed values of  $(\mathbf{E}_S, p_f, \theta)$  the quantity  $\delta$  has a minimum in the equilibrium state. The necessary condition for an extremum  $\partial\delta/\partial\boldsymbol{\gamma}^R = 0$  when  $\boldsymbol{\gamma} = \mathbf{w} = 0$  leads to equality (3.17). The condition  $\partial\delta/\partial\mathbf{w}^R = 0$  is satisfied identically by virtue of definition (2.8) of the dissipated interaction forces.

Expressions for the second derivatives at the point  $\boldsymbol{\gamma} = \mathbf{w} = 0$  lead to symmetry of the tensors  $\mathbf{Y}_\gamma$  and  $\mathbf{Y}_w$ . The non-negativity of the second differential of the dissipation gives properties (3.18).

**Corollary 2.** *The rate of change of the entropy density for smooth flows is determined by the rate of supply of heat and the seepage dissipation*

$$\rho_S \theta \frac{d_S \eta_S}{dt} + \rho_F \theta \frac{d_F \eta_F}{dt} = \nabla \cdot \mathbf{q} + \rho r + \mathbf{w}^R \cdot \mathbf{b}^\times \tag{3.19}$$

Equality (3.19) follows directly from formulae (3.6)–(3.8). It can be seen that the adiabatic process ( $\mathbf{q} = 0, \rho r = 0$ ) for a thermoelastic porous solid is not an isoentropic process (the Joule-Thomson effect).

Relation (3.19) serves as an additional conservation law, which can be used for symmetrizing the system of equations.<sup>11,15–16</sup>

**Corollary 3.** *Suppose the skeleton is an initially isotropic rigid body with an undistorted reference configuration  $\kappa_0(t)$ , the symmetry group in which  $g_0 = o$  for all instants of time. To satisfy the principles of objectivity and thermodynamic consistency it is necessary and sufficient that the governing equations of the porous medium with initially isotropic skeleton should have the form*

$$\begin{aligned} \Psi_F &= \Psi_F(\rho_f, \theta), \quad p_f = \rho_f^2 \frac{\partial \Psi_F}{\partial \rho_f}, \quad \eta_F = -\frac{\partial \Psi_F}{\partial \theta} \\ \Psi &= \Psi(I_k(\mathbf{U}_S), p_f, \theta), \quad \boldsymbol{\sigma} = -\rho_S \frac{\partial \Psi}{\partial \mathbf{G}_S^T} \cdot \mathbf{G}_S, \quad \eta_S = -\frac{\partial \Psi}{\partial \theta}, \quad \phi = -\rho_S \frac{\partial \Psi}{\partial p_f} \\ \mathbf{q} &= \mathbf{q}(\mathbf{U}_S, \rho_f, \theta, \boldsymbol{\gamma}, \mathbf{w}), \quad \mathbf{b}^{\text{dis}} = \mathbf{b}(\mathbf{U}_S, \rho_f, \theta, \boldsymbol{\gamma}, \mathbf{w}) \end{aligned} \tag{3.20}$$

where  $I_k(\mathbf{U}_S)$  are independent invariants of the tensor  $\mathbf{U}_S$ ,  $\Psi$  is a scalar isotropic function of  $\mathbf{q}$ , and  $\mathbf{b}$  are isotropic vector functions of these arguments. The isotropy of the functions  $\Psi, \mathbf{q}, \mathbf{b}$  indicates that for any orthogonal tensor  $\mathbf{Q}(t)$

$$\begin{aligned} \Psi(\mathbf{U}_S, p_f, \theta) &= \Psi(\mathbf{Q} \cdot \mathbf{U}_S \cdot \mathbf{Q}^T, p_f, \theta) \\ \mathbf{Q} \cdot \mathbf{q}(\mathbf{U}_S, \rho_f, \theta, \boldsymbol{\gamma}, \mathbf{w}) &= \mathbf{q}(\mathbf{Q} \cdot \mathbf{U}_S \cdot \mathbf{Q}^T, \rho_f, \theta, \mathbf{Q} \cdot \boldsymbol{\gamma}, \mathbf{Q} \cdot \mathbf{w}) \\ \mathbf{Q} \cdot \mathbf{b}(\mathbf{U}_S, \rho_f, \theta, \boldsymbol{\gamma}, \mathbf{w}) &= \mathbf{b}(\mathbf{Q} \cdot \mathbf{U}_S \cdot \mathbf{Q}^T, \rho_f, \theta, \mathbf{Q} \cdot \boldsymbol{\gamma}, \mathbf{Q} \cdot \mathbf{w}) \end{aligned} \tag{3.21}$$

To prove relations (3.20) and (3.21) we turn to the governing relations in the form (3.6). We fix the point  $(\mathbf{x}, t)$ . We take the tensor  $\mathbf{P}_S = \mathbf{R}_S^T(\mathbf{x}, t) \in 0$  as an element of the symmetry group of the skeleton, where  $\mathbf{R}_S$  is the rotation tensor, which occurs in the polar expansion (1.2). For such a transformation of the undistorted reference configuration  $\kappa_0$  the distortion  $\mathbf{G}_S \rightarrow \mathbf{U}_S$ , the rotation tensor  $\mathbf{R}_S \rightarrow \mathbf{I}$ , while the vectors  $\mathbf{R}_S \cdot \boldsymbol{\gamma} \rightarrow \boldsymbol{\gamma}, \mathbf{R}_S \cdot \mathbf{w} \rightarrow \mathbf{w}$ . Taking into account the invariance of the functions  $\psi_A(\Lambda), \mathbf{q}(\Lambda)$  and  $\mathbf{b}(\Lambda)$ , where  $\Lambda = (\mathbf{G}_S, \rho_f, \theta)$ , with reference to the orthogonal transformation of the configuration  $\kappa_0$  we obtain

$$\begin{aligned} \psi_S &= \psi_S(\mathbf{G}_S, \rho_f, \theta) = \Psi(\mathbf{U}_S, \rho_f, \theta) \\ \mathbf{q} &= \mathbf{R}_S^T \cdot \mathbf{q}^\times(\mathbf{G}_S, \rho_f, \theta, \mathbf{R}_S \cdot \boldsymbol{\gamma}, \mathbf{R}_S \cdot \mathbf{w}) = \mathbf{q}(\mathbf{U}_S, \rho_f, \theta, \boldsymbol{\gamma}, \mathbf{w}) \\ \mathbf{b}^{\text{dis}} &= \mathbf{R}_S^T \cdot \mathbf{b}^\times(\mathbf{G}_S, \rho_f, \theta, \mathbf{R}_S \cdot \boldsymbol{\gamma}, \mathbf{R}_S \cdot \mathbf{w}) = \mathbf{b}(\mathbf{U}_S, \rho_f, \theta, \boldsymbol{\gamma}, \mathbf{w}) \end{aligned}$$

Taking formulae (3.4) into account we arrive at the necessity of relations (3.20).



The sufficiency of the form (3.20) for invariance with respect to simultaneous orthogonal transformations of the undistorted reference configuration of the skeleton, unimodular transformations of the reference configuration of the fluid and orthogonal transformations of the current configuration of the body is obvious if we take into account how the arguments of the functions (3.20) and properties (3.21) vary for such transformations.

In applied investigations it is sometimes more convenient to use the Almansi tensor, defined by relation (1.3). In this case the governing equations of the initially isotropic skeleton can be written as

$$\Psi = \Psi(I_k(\mathbf{A}_A), p_f, \theta), \quad \boldsymbol{\sigma} = \rho_S(\mathbf{I} - 2\mathbf{A}_S) \cdot \frac{\partial \Psi}{\partial \mathbf{A}_S}, \quad \eta_S = -\frac{\partial \Psi}{\partial \theta}, \quad \phi = -\rho_S \frac{\partial \Psi}{\partial p_f}$$

Hence it follows that the total-stress tensor is an isotropic function of the symmetric tensor  $\mathbf{A}_S$ , which can be written in the form of a second-degree polynomial

$$\boldsymbol{\sigma} = \beta_0 \mathbf{I} + \beta_1 \mathbf{A}_S + \beta_2 \mathbf{A}_S^2$$

where the coefficients  $\beta_i = \beta_i(I_k(\mathbf{A}_S), p_f, \theta)$  are determined by the derivatives of the potential  $\Psi(I_k(\mathbf{A}_S), p_f, \theta)$ . This can be shown using the definition of the principal invariants  $I_k(\mathbf{A}_S)$  and the formulae for their derivatives

$$\frac{\partial I_1}{\partial \mathbf{A}_S} = \mathbf{I}, \quad \frac{\partial I_2}{\partial \mathbf{A}_S} = I_1 \mathbf{I} - \mathbf{A}_S, \quad \frac{\partial I_3}{\partial \mathbf{A}_S} = \mathbf{A}_S^2 - I_1 \mathbf{A}_S + I_2 \mathbf{I}$$

#### 4. A thermoelastic porous incompressible skeleton

We will consider the consequences of the principles of objectivity and thermodynamic consistency for a porous medium with an incompressible skeleton ( $\rho_s = \text{const}$ ). In this case from the law of conservation of mass (2.1) and the formulae

$$\rho_S = (1 - \phi)\rho_s, \quad \rho_{\kappa_S} = (1 - \phi_0)\rho_s$$

where  $\phi$  and  $\phi_0$  are the actual and initial value of the porosity, we obtain

$$1 - \phi = (1 - \phi_0)J_S, \quad J_S = \det \mathbf{G}_S \tag{4.1}$$

i.e. the porosity is determined by the distortion of the skeleton. Hence, the governing equations of this medium do not contain an independent relation for the porosity.

The form of the governing relations of a thermoelastic porous medium with an incompressible skeleton, necessary and sufficient to satisfy these principles, has the form

$$\begin{aligned} \Psi_F &= \Psi_F(\rho_f, \theta), \quad p_f = \rho_f^2 \frac{\partial \Psi_F}{\partial \rho_f}, \quad \eta_F = -\frac{\partial \Psi_F}{\partial \theta} \\ \Psi_S &= \Psi_S(\mathbf{V}_S, \theta), \quad \boldsymbol{\sigma}^{\text{eff}} = -\rho_S \frac{\partial \Psi_S}{\partial \mathbf{G}_S^T} \cdot \mathbf{G}_S, \quad \eta_S = -\frac{\partial \Psi_S}{\partial \theta} \end{aligned} \tag{4.2}$$

$$\mathbf{q} = \mathbf{R}_S^T \cdot \mathbf{q}^\times(\mathbf{V}_S, \rho_f, \theta, \boldsymbol{\gamma}^R, \mathbf{w}^R), \quad \mathbf{b}^{\text{dis}} = \mathbf{R}_S^T \cdot \mathbf{b}^\times(\mathbf{V}_S, \rho_f, \theta, \boldsymbol{\gamma}^R, \mathbf{w}^R), \quad \delta_f + \delta_T \geq 0$$

The most important feature of these equations is the fact that the free energy of the skeleton is independent of the true density of the fluid and defines the effective stress tensor

$$\boldsymbol{\sigma}^{\text{eff}} = \boldsymbol{\sigma} + p_f \mathbf{I} = (1 - \phi)(\boldsymbol{\sigma}_s + p_f \mathbf{I})$$

The idea of an effective stress, which goes back to the book by Terzaghi<sup>8</sup> and has been considered by numerous authors,<sup>9</sup> is widely used in soil mechanics, the petroleum business, when modelling concretes and rocks under moderate loads, when the compressibility of the skeleton can be neglected. Relations (4.2) are unusual in that the effective stresses

in the model considered are only produced by the requirement of thermodynamic consistency, without invoking any additional assumptions.

To prove relations (4.2) we will use the fact that the saturating fluid possesses a unimodular symmetry group. In conjunction with the principle of objectivity this leads to relations that are necessary and sufficient to satisfy this principle:

$$\begin{aligned} \psi_A &= \psi_A^\times(\Lambda^R), \quad \sigma_A = \mathbf{R}_S^T \cdot \sigma_A^\times(\Lambda^R) \cdot \mathbf{R}_S, \quad \eta_A = \eta_A^\times(\Lambda^R), \quad \mathbf{q} = \mathbf{R}_S^T \cdot \mathbf{q}^\times(\Lambda^R) \\ \mathbf{b}^{\text{dis}} &= \mathbf{R}_S^T \cdot \mathbf{b}^\times(\Lambda^R), \quad \Lambda^R \equiv (\mathbf{V}_S, \rho_f, \theta, \boldsymbol{\gamma}^R, \mathbf{w}^R), \quad \boldsymbol{\gamma}^R = \mathbf{R}_S \cdot \boldsymbol{\gamma}, \quad \mathbf{w}^R = \mathbf{R}_S \cdot \mathbf{w} \end{aligned} \tag{4.3}$$

In order to clarify the limitations imposed by the second law of thermodynamics, we will introduce, as before, the equilibrium state. Suppose  $\mathbf{b}^0 = -\sigma_f^0 \cdot \nabla \phi(J_S)$  is the equilibrium interaction force,  $\sigma_f^0$  is the equilibrium stress tensor in the fluid, and  $\sigma_F^{\text{dis}}, \mathbf{b}^{\text{dis}}$  are the dissipative stress in the fluid and the dissipative interaction force, equal to zero in the equilibrium state. Taking relation (4.1) into account, it follows from continuity equation (2.2) that

$$\rho_f^{-1} \phi \frac{d_F \rho_f}{dt} - \mathbf{G}_S^{-1} : \frac{d_S \mathbf{G}_S^T}{dt} + \nabla \cdot (\phi \mathbf{w}) = 0 \tag{4.4}$$

Eqs. (4.4) and (3.10) differ considerably: first, in the expression for the divergence of the rate of seepage in the case of an incompressible skeleton, there is no derivative with respect to time of the porosity of the medium; second, the factor, equal to the porosity of the medium in front of the derivative of the distortion of the skeleton disappears. It is these features that produce the effective stress tensor and which are responsible for the fact that there is no dependence of the energy of the skeleton on the fluid density or the porosity of the medium. In fact, taking relation (4.4) for an incompressible skeleton into account, we obtain

$$\sigma_f^0 : (\nabla \otimes (\phi \mathbf{w})) = -p_f^0 \left( \mathbf{G}_S^{-1} : \frac{d_S \mathbf{G}_S^{-1}}{dt} - \rho_f^{-1} \phi \frac{d_F \rho_f}{dt} \right) + \mathbf{s}_f^0 : (\nabla \otimes (\phi \mathbf{w}))$$

and inequality (2.14) for the rate of entropy production can be written in the form

$$\begin{aligned} \sum \rho_A \left( \frac{d_A \psi_A}{dt} + \eta_A \frac{d_A \theta}{dt} \right) - (\sigma^{\text{eff}} \cdot \mathbf{G}_S^{-1}) : \frac{d_S \mathbf{G}_S^T}{dt} + \frac{\phi p_f^0 d_F \rho_f}{\rho_f dt} + \\ + \mathbf{s}_f^0 : (\nabla \otimes (\phi \mathbf{w})) + \sigma_F^{\text{eff}} : (\nabla \otimes \mathbf{w}) - \mathbf{b}^{\text{dis}} \cdot \mathbf{w} + \delta_T \geq 0 \end{aligned} \tag{4.5}$$

Consider the state which varies linearly in the neighbourhood of the point  $(\mathbf{x}_0, t_0)$

$$\Lambda(\mathbf{x}, t) = \Lambda^0 + a(t - t_0) + (\mathbf{x} - \mathbf{x}_0) \cdot \mathbf{B}, \quad t \geq t_0$$

where  $a$  and  $\mathbf{B}$  are the rates of variation and gradients of the parameters of state of the medium at the point  $(\mathbf{x}_0, t_0)$ . Calculating the derivatives  $d_A \psi_A / dt$  and substituting them into inequality (4.5), we obtain

$$\frac{\partial \psi_A}{\partial \mathbf{w}} = 0, \quad \frac{\partial \psi_A}{\partial \boldsymbol{\gamma}} = 0, \quad \mathbf{s}_f^0 = 0, \quad \sigma_F^{\text{dis}} = 0 \tag{4.6}$$

$$\frac{\partial \psi_F}{\partial \mathbf{G}_S^T} = 0, \quad p_f = \rho_f^2 \frac{\partial \psi_F}{\partial \rho_f}, \quad \eta_F = -\frac{\partial \psi_F}{\partial \theta} \tag{4.7}$$

$$\sigma^{\text{eff}} = -\rho_S \frac{\partial \psi_S}{\partial \mathbf{G}_S^T} \cdot \mathbf{G}_S, \quad \eta_S = -\frac{\partial \psi_S}{\partial \theta}, \quad \frac{\partial \psi_S}{\partial \rho_f} = 0$$

$$\delta_f + \delta_T \geq 0, \quad \delta_f = \mathbf{b}^{\text{dis}} \cdot \mathbf{w}, \quad \delta_T = \theta^{-1} \mathbf{q} \cdot \boldsymbol{\gamma} \tag{4.8}$$

It can be seen from Eq. (4.6) that, as for a compressible material, the free energy of the components of a thermoelastic saturated porous medium is independent of the temperature gradient and the relative velocity, while the tensor of the

dissipative stresses and the deviator of the equilibrium stress in the fluid are equal to zero. Hence, the stress in the fluid is a spherical tensor  $\sigma_F = -p_F^0 \mathbf{I}$ .

The first three equalities of (4.7) show that the energy of the fluid is independent of  $\mathbf{G}_S$ , while the pore pressure and the entropy of the fluid are determined by the derivatives of the function  $\psi_F = \psi_F(\rho_f, \theta)$ . Since the energy is independent of  $\boldsymbol{\gamma}$  and  $\mathbf{w}$ , the difference between  $p_f^0$  and  $p_f$  disappears. It follows from the last three equations of (4.7) that the energy of the skeleton is independent of the fluid density  $\rho_f$ , and the tensor of the effective stresses and the entropy of the skeleton are determined by the derivative of the energy  $\psi_S = \psi_S(\mathbf{G}_S, \theta)$ .

Hence, we have shown the necessity of relations (4.2).

Conversely, if relations (4.2) are satisfied for any state  $\Lambda(t)$ , this is sufficient to satisfy inequality (4.5).

The case of an incompressible skeleton, saturated with an incompressible fluid is of undoubted interest for applications. In this case  $\rho_s = \text{const}$  and  $\rho_f = \text{const}$ . From the continuity equations for the skeleton and the fluid, taking the kinematic relation (1.4) into account, we obtain

$$\nabla \cdot (\phi \mathbf{w}) - \mathbf{G}_S^{-1} : \frac{d_S \mathbf{G}_S^T}{dt} = 0 \tag{4.9}$$

The form of the governing relations, necessary and sufficient to satisfy the principles of objectivity and thermodynamic consistency, has the form

$$\begin{aligned} \psi_F &= \psi_F(\theta), \quad \eta_f = -\frac{\partial \psi_F}{\partial \rho_f}, \quad \sigma_F = -p_F \mathbf{I} \\ \psi_S &= \psi_S(\mathbf{V}_S, \theta), \quad \sigma^{\text{eff}} = -\rho_s \frac{\partial \psi_S}{\partial \mathbf{G}_S^T} \cdot \mathbf{G}_S, \quad \eta_s = -\frac{\partial \psi_S}{\partial \theta} \end{aligned} \tag{4.10}$$

$$\mathbf{q} = \mathbf{R}_S^T \cdot \mathbf{q}^\times(\mathbf{V}_S, \theta, \boldsymbol{\gamma}^R, \mathbf{w}^R), \quad \mathbf{b}^{\text{dis}} = \mathbf{R}_S^T \cdot \mathbf{b}^\times(\mathbf{V}_S, \theta, \boldsymbol{\gamma}^R, \mathbf{w}^R), \quad \delta_f + \delta_T \geq 0$$

where  $p_F$  is a scalar parameter, which is determined by the boundary conditions of the problem.

In fact, the arguments of the governing relations of such a medium are the quantities  $(\mathbf{G}_S, \theta, \boldsymbol{\gamma}$  and  $\mathbf{w})$ . The proof rests on the definition of the fluid, supplemented by the condition  $\rho_s = \text{const}$  and the connection (4.1) of the porosity with the distortion of the skeleton. The principle of objectivity gives

$$\psi_A = \psi_A^\times(\Lambda^R), \quad \mathbf{q} = \mathbf{R}_S^T \cdot \mathbf{q}^\times(\Lambda^R), \quad \mathbf{b}^{\text{dis}} = \mathbf{R}_S^T \cdot \mathbf{b}^\times(\Lambda^R), \quad \Lambda^R \equiv (\mathbf{V}_S, \theta, \boldsymbol{\gamma}^R, \mathbf{w}^R)$$

Inequality (2.14) for the rate of entropy production, taking Eq. (4.9) into account, can be written in the form

$$-\sum \left( \rho_A \frac{d_A \psi_A}{dt} + \rho_A \eta_A \frac{d_A \theta}{dt} \right) - (\sigma^{\text{eff}} \cdot \mathbf{G}_S^{-1}) : \frac{d_S \mathbf{G}_S^T}{dt} + s_f^0 : (\nabla \otimes (\phi \mathbf{w})) + \delta_f + \delta_T \geq 0$$

Substituting here the expression for the derivative  $d_A \psi_A(\mathbf{G}_S, \theta, \boldsymbol{\gamma}, \mathbf{w})/dt$ , we obtain

$$\begin{aligned} & - \left( \rho_s \frac{\partial \psi_S}{\partial \mathbf{G}_S^T} + \sigma^{\text{eff}} \cdot \mathbf{G}_S^{-1} \right) : \frac{d_S \mathbf{G}_S^T}{dt} - \rho_F \frac{\partial \psi_F}{\partial \mathbf{G}_S^T} : \frac{d_F \mathbf{G}_S^T}{dt} - \sum \rho_A \frac{\partial \psi_A}{\partial \mathbf{w}} \cdot \frac{d_A \mathbf{w}}{dt} - \sum \rho_A \frac{\partial \psi_A}{\partial \boldsymbol{\gamma}} \cdot \frac{d_A \boldsymbol{\gamma}}{dt} - \\ & - \sum \rho_A \left( \frac{\partial \psi_A}{\partial \theta} + \eta_A \right) \frac{d_A \theta}{dt} + s_f^0 : (\nabla \otimes (\phi \mathbf{w})) + \sigma_F^{\text{dis}} : (\nabla \otimes \mathbf{w}) + \mathbf{b}^{\text{dis}} \cdot \mathbf{w} + \delta_T \geq 0 \end{aligned}$$

Hence it follows that the free energy of the components is independent of the temperature gradient and the relative velocity. The dissipative-stress tensor and the deviator of the equilibrium stress of the fluid are equal to zero, so that  $\sigma_F = -p_F \mathbf{I}$ . The energy of the fluid is independent of the distortion of the skeleton, while the entropy of the fluid is determined by the derivative of the function  $\psi_F = \psi_F(\theta)$ . The effective stress tensor and the entropy of the skeleton are determined by the partial derivative of the free energy  $\psi_S = \psi_S(\mathbf{G}_S, \theta)$ . The remaining inequality is equivalent to the non-negativity of the total dissipation. The inequality for the rate of entropy production imposes no limitations on the

pore pressure of an incompressible fluid, saturating an incompressible skeleton. This indicates that pore pressure is not determined by the actual deformation and temperature. The necessity of relations (4.10) is demonstrated.

Conversely, if relations (4.10) are satisfied for any state  $\Lambda(t)$ , this is sufficient to satisfy inequality (2.14) with condition (4.9).

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